where ℓ_{1x}, ℓ_{1y}, ℓ_{1z} are the dimensions of the object of thermostating in the directions of the axes of coordinates; t_0, t_2 are the temperatures of the light window and of the chamber, respectively, obtained at the preceding stages after the problems (14)-(19) and (20)-(23) had been solved.

In accordance with (25) the object takes part in symmetric heat exchange with the light windows along the z-axis (25a), and along the other two axes it is in ideal contact with the chamber (25b). Condition (25c) closes the system.

The presented example shows that the described approach makes it possible from the same positions to arrive at the synthesis of the design of thermostats for different objects.

A further task is to work out actual algorithms for choosing the design parameters satisfying the requirements of the technical specification.

NOTATION

t_i, t_j, temperature of the i-th and j-th element, respectively, of the thermostat; t_{Vi}, t_{Si}, mean volumetric and mean surface temperature, respectively, of the i-th element of the thermostat; C_i, c_i, ρ_i, λ_i, full and specific heat capacity, density, and thermal conductivity, respectively, of the i-th element of the thermostat; q_{Vi}, q_{Si}, specific power of the volumetric and surface heat sources, respectively; τ, time; $S_i(r)$, running area of the isothermal surface; r, generalized coordinate; σ_{ij}, thermal conductivity between the i-th and j-th elements of the thermostat; α_{Vi}, volumetric heat-transfer coefficient with the inner convective medium; α_{N2}, heat-transfer coefficient of the element of the thermostat with the environment; u_{Vi}, temperature of the inner convective medium; $q_{Si,1}$, $q_{Si,2}$, specific power of the surface heat sources on the inner and outer surface, respectively, of the i-th jacket; P_i, full power of the heat sources in the i-th element; i, j, subscripts denoting the ordinal numbers of the elements of the thermostat.

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PARAMETRIC METHOD FOR THE SOLUTION OF AN ILL-POSED INVERSE HEAT-CONDUCTION PROBLEM IN APPLICATION TO THE OPTIMIZATION OF THERMAL REGIMES

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A method is proposed for the stable approximate solution of an ill-posed inverse heat-conduction problem, to which the investigated problem of optimal control of the thermal regime of a rigid body is reduced.

The timeliness of nondestructive testing problems and the difficulties of reconstructing the temperature field and the thermophysical characteristics of an object from experimental results have fostered the rapid development of identification methods in heat conduction [1-3]. It has been shown [4] that a number of problems in the control of the thermal regime of a rigid body can also be solved by reducing them to an inverse heat-conduction problem (IHCP). In the latter case, as a rule, certain characteristic (thermal or thermomechanical)

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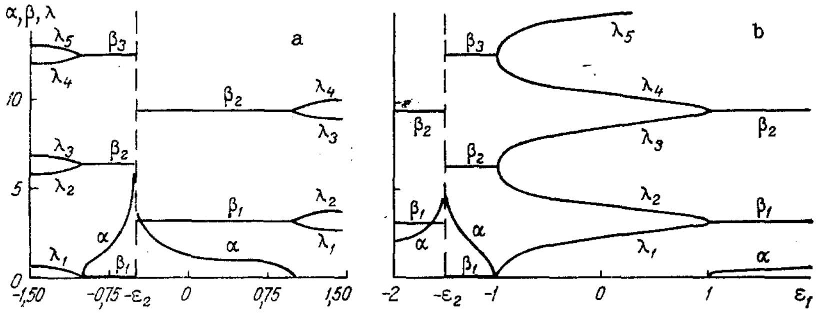

Fig. 1. Roots μ_n of the characteristic equation (13) vs ε_1 ($\mu n = \alpha_n \pm i\beta_n$, $\mu_n = i\lambda_n$, $i = \sqrt{-1}$, $n = 1, 2, \ldots$): a) $\varepsilon_2 = 0.5$; b) 1.5.

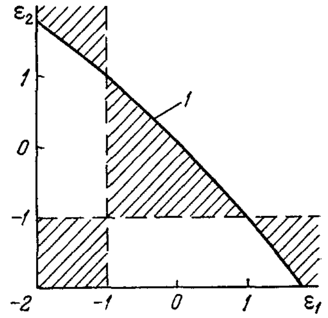

Fig. 2. Stability of the solution (11), (12) according to the values of the parameters $\varepsilon_{1,2}$. 1) Solution of Eq. (13) for $\alpha = \beta_1 = \pi$.

variables of the process are specified, and the object of solving the problem is to determine the conditions under which the temperature field corresponding to such characteristics can be established. In the present study, elaborating the results set forth in [5], we propose a method that can be used to solve one such control problem for a one-dimensional temperature regime.

A number of processes of heat treatment of materials require that a certain surface $x = k$, often inaccessible for direct treatment, be maintained at a definite temperature $\varphi(\tau)$, which is specified within allowed technological error limits Δ_T:

$$\max_{\tau \in (0,\, \tau_0]} |T(k,\, \tau) - \varphi(\tau)| \leqslant \Delta_T\,. \qquad (19$$

Suppose that the heat-treatment process is controlled by convective heat transfer on the opposite surface $x = 1$:

$$\frac{\partial T(1,\, \tau)}{\partial x} + H[T(1,\, \tau) - u(\tau)] = 0. \qquad (2)$$

The problem is to control the one-dimensional temperature regime $T(x, \tau)$ in a specified time interval $\tau \in [0, \tau_0]$ by means of the temperature $u(\tau)$ of the warming medium so as to guarantee the heating performance factor (optimality criterion) (1). The temperature field in the body in this case satisfies the heat-conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{j}{x}\,\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \tau}\ (j = 0,\ 1,\ 2), \qquad (3)$$

$$\tau > 0,\ x \in (k,\ 1),\ k = \frac{r_1}{r_2} \in [0,\ 1),$$

the initial condition

$$T(x,\ 0) = f(x),\ x \in [k,\ 1], \qquad (4)$$

and the boundary condition

$$\frac{\partial T(k,\, \tau)}{\partial x} = \psi(\tau), \qquad (5)$$

and the control function $u(\tau)$ is subjected to the constraints

$$t_1(\tau) \leqslant u(\tau) \leqslant t_2(\tau). \tag{6}$$

Here j = 0, 1, 2 for a plate (k = 0), a cylinder, and a sphere, respectively.

We replace the real heating performance condition (1) by the idealized version

$$T(k, \tau) = \varphi(\tau). \tag{7}$$

Then, following the procedure developed in [4] and determining the temperature field $T(x, \tau)$ satisfying the IHCP (3)-(5), (7), from the heat-transfer condition (2) we readily find a control function $u(\tau)$ that will ensure the satisfaction of condition (7), i.e., the objective function (1) for $\Delta_T = 0$.

The IHCP (3)-(5), (7), which belongs to the class of boundary-value [1] or outer [3] IHCP's, is known to be an ill-posed problem owing to the absence of a bounded iverse operator [6]. We note that the exact value of the function $\varphi(\tau)$ in the given problem can belong to the set of functions that are unrealizable under the given boundary conditions (4), (5).

Condition (7) has been modified somewhat [5] in order to regularize the solution of such an IHCP. However, for sufficiently small values of the regularization parameter the amplitude of the oscillations of this solution grows considerably, and this can violate the control constraints (6). With the latter consideration in mind, we alter both boundary conditions specified on the treated surface, as follows:

$$T(k, \tau) + \varepsilon_1 R_1(T) = \varphi(\tau), \tag{8}$$

$$\frac{\partial T(k, \tau)}{\partial x} + \varepsilon_2 R_2(T) = \psi(\tau), \tag{9}$$

where the operators R_i (i = 1, 2) can have the form [5]

$$\begin{aligned} &\text{a)}\quad R(T) = T(1, \tau) - f(1);\\ &\text{b)}\quad R(T) = \frac{\partial T(1, \tau)}{\partial x};\\ &\text{c)}\quad R(T) = \frac{1+j}{1-k^{1+j}} \int_k^1 x^j [T(x, \tau) - f(x)]\, dx. \end{aligned} \tag{10}$$

It can be verified that the problem (3), (4),(8), (9) has a unique solution [4, 7]. The Laplace integral transform [8, 9] is used to represent the solution in the form [4]

$$T_\varepsilon(x, \tau) = \frac{\partial}{\partial \tau} \int_0^\tau [\varphi_R(\tau-\eta)\, T_1(x, \eta) + \psi_R(\tau-\eta)\, T_2(x, \eta)]\, d\eta + \int_k^1 \xi^j f(\xi)\, T_3(x, \xi, \tau)\, d\xi. \tag{11}$$

Here the form of the functions $T_{1,2,3}$, φ_R, and ψ_R depends on the choice of the operators R_i in conditions (8) and (9). For example, in the case of a plate (j = k = 0) with the operators R_1 and R_2 chosen to be of the type (10a) and (10b), respectively, the influence functions are

$$T_1(x, \tau) = \frac{1}{1+\varepsilon_1} + \frac{4}{\varepsilon_1+\varepsilon_2} \operatorname{Re} \sum_{n=1}^\infty \frac{\operatorname{ch} \mu_n x + \varepsilon_2 \operatorname{ch} \mu_n (1-x)}{\mu_n \operatorname{sh} \mu_n} \exp(\mu_n^2 \tau),$$

$$T_2(x, \tau) = \frac{x(1+\varepsilon_1) - \varepsilon_1}{(1+\varepsilon_1)(1+\varepsilon_2)} + \frac{4}{\varepsilon_1+\varepsilon_2} \operatorname{Re} \sum_{n=1}^\infty \frac{\operatorname{sh} \mu_n x - \varepsilon_1 \operatorname{sh} \mu_n (1-x)}{\mu_n^2 \operatorname{sh} \mu_n} \exp(\mu_n^2 \tau),$$

$$T_3(x, \xi, \tau) = \frac{4\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2} \operatorname{Re} \sum_{n=1}^\infty \frac{\operatorname{sh} \mu_n (x-\xi)}{\operatorname{sh}\mu_m} \exp(\mu_n^2 \tau) + \tag{12}$$

$$+ \frac{4}{\varepsilon_1+\varepsilon_2} \operatorname{Re} \sum_{n=1}^\infty \frac{\varepsilon_1 \operatorname{sh}\mu_n (1-\xi) \operatorname{ch} \mu_n x + \varepsilon_2 \operatorname{ch} \mu_n (1-\xi) \operatorname{sh} \mu_n x}{\operatorname{sh} \mu_n} \exp(\mu_n^2 \tau),$$

and the functions $\varphi_R(\tau) = \varphi(\tau) + \varepsilon_1 f(1)$; $\psi_R(\tau) = \psi(\tau)$. Here $\mu_n = \alpha_n \pm i\beta_n (\alpha_n, \beta_n \geq 0$; $i = \sqrt{-1}$; $n = 1, 2, \ldots)$ is the countable set of complex roots of the characteristic equation

$$1 + (\varepsilon_1 + \varepsilon_2) \operatorname{ch} \mu + \varepsilon_1 \varepsilon_2 = 0. \tag{13}$$

We analyze the stability of the solution (11), (12) by means of the spectrum of eigenvalues of the heat-conduction problem, i.e., the roots of its characteristic equation (13). The resulting solution will be stable if the values of all the roots μ_n satisfy the condition $\alpha_n \leq \beta_n$ [8]. An analysis of the roots of Eq. (13) shows that $\alpha_n = \alpha$ $(n = 1, 2, \ldots)$, and so the satisfaction of the condition $\alpha \leq \beta_1$ is sufficient in order for the solution to be stable for $\beta_1 < \beta_2 < \ldots < \beta_n < \ldots$.

We now consider the behavior of the solution (11), (12) as a function of the possible choices of the parameters ε_i $(i = 1, 2)$. Figure 1 shows the values of the roots of the characteristic equation of the problem with variation of the quantity ε_1. We note that the variation of the roots as a function of ε_2 is similar, owing to the symmetry of Eq. (13) with respect to the parameters. It is evident from the graphs that the real parts of the roots of Eq. (13) grow without bound if the values of ε_i are chosen to be close in absolute value and of opposite sign, and also in this case

$$\lim_{\varepsilon_1+\varepsilon_2 \to 0} \alpha = \infty,$$

indicating the absence of a classical continuous solution for the problem (3), (4), (8), (9) in this situation. Consequently, the choice of parameters $\varepsilon_1 = -\varepsilon_2$ is unacceptable. Also inadmissible is the situation where at least one of the parameters is $\varepsilon_i = -1$, since the existence of the root zero of Eq. (13) causes the influence functions $T_{1,2}$ to have the form

$$T_i(x, \tau) = a_i(x) + b_i(x)\tau + 2 \operatorname{Re} \sum_{n=1}^{\infty} \frac{A_i(x, \mu_n) + \tau B_i(x, \mu_n)}{\exp^{-1}(\mu_n^2 \tau)},$$

where a_i, b_i, A_i, and B_i are known functions. Consequently, $\lim\limits_{\tau\to\infty} T_i = \infty$ $(i = 1, 2)$ for $\varepsilon_1 = -1$, and $\lim\limits_{\tau\to\infty} T_2 = \infty$; $b_i = 0$ for $\varepsilon_2 = -1$, making it impossible to satisfy the optimality criterion (1).

We denote

$$A(\varepsilon) = \frac{1 + \varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}.$$

It can be shown on the basis of Eq. (13) that for any pair of values $|\varepsilon_1| < 1$, $|\varepsilon_2| > 1$ and $|\varepsilon_1| > 1$, $|\varepsilon_2| < 1$, corresponding to the condition $-1 < A(\varepsilon) < 1$, the real part of all the roots $\alpha_n = \alpha = 0$, and the imaginary part $\lambda_n > 0$, $n = 1, 2, \ldots$. Consequently, the solution (11), (12), has a stable exponential character for such a choice of the parameters ε_j, and also for the case where at least one of them $\varepsilon_j = 1$, $j = 1, 2$ $[A(\varepsilon) = 1]$ and the roots $\mu_n = i\lambda_n$ are two-valued.

In cases where: a) $|\varepsilon_j| < 1$ $(j = 1, 2)$, $\varepsilon_1 + \varepsilon_2 > 0$ (Fig. 1a); b) $\varepsilon_j > 1$ (Fig. 1b); c) $|\varepsilon_j| > 1$, $\varepsilon_1 + \varepsilon_2 < 0$, $\operatorname{sign}\varepsilon_1 \neq \operatorname{sign}\varepsilon_2$ (Fig. 1b), the roots $\mu_n = \alpha_n \pm i\beta_n$ are complex valued $[\alpha > 0$, $\beta_n = (2n-1)\pi$, $n = 1, 2, \ldots]$, and the solution $\tau_\varepsilon(x, \tau)$ will be stable under the condition $\alpha \leq \beta_1$. Straightforward transformations of Eq. (13) can be used for $\beta_1 = \pi$ to replace this condition by its equivalent $1 < A(\varepsilon) \leq \cosh\pi$. If the latter condition is satisfied, the solution (11), (12) will be stable and oscillate with a decaying amplitude.

For all choices of the parameters ε_j other than the above-described combinations, the real and imaginary parts of the roots μ_n are equal to $\alpha_n = \alpha > 0$, $\beta_n = 2\pi(n-1)$, $n = 1, 2, \ldots$ The first root $\mu_1 = \alpha$ is real, and so the condition $\alpha \leq \beta_1$ is not satisfied, making the solution unstable.

The hatched regions in Fig. 2 show where the solution $T_\varepsilon(x, \tau)$ is unstable if the values of the parameters ε_j $(j = 1, 2)$ are chosen in those regions.

It follows from the foregoing analysis of the spectrum of eigenvalues of Eq. (13) that the condition for stability of the solution (11), (12) can be written in the form

$$-1 < A(\varepsilon) \leqslant \operatorname{ch}\pi. \tag{14}$$

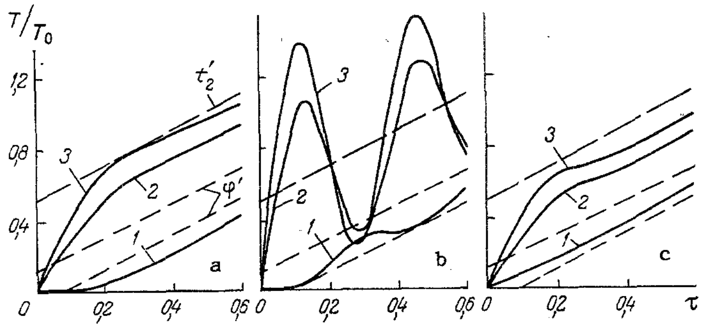

Fig. 3. Solution of the thermal regime control problem for a plate. 1) Relative temperature of treated surface $T_\varepsilon(0, \tau)/T_0$; 2) relative temperature of heated surface $T_\varepsilon(1, \tau)/T_0$; 3) control function $u_\varepsilon(\tau)/T_0$; $t_2'(\tau) = t_2(\tau)/T_0$; $\varphi'(\tau) = \varphi(\tau)/T_0 \pm \Delta'_T$; $\Delta'_T = \Delta_T/T_0 = 0.1\ (-0.4\ \tau)$; $H = 8$; $\omega' = \omega/T_0$ (all dimensionless quantities). a) $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0$; b) 0.1, 0; c) 0.1, 0.25.

We denote the stability domain by Ω_{14}, where Ω_q is the domain of definition of the pair (ε_1, ε_2) subject to condition (q). Then the solution is stable for a pair of parameters corresponding to a point $(\varepsilon_1, \varepsilon_2) \in \Omega_{14}$ in the part of the coordinate plane shown in Fig. 2.

It can be shown on the basis of the solution (11), (12) that

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} T_\varepsilon(0, \tau) = \varphi(\tau).$$

Since the solution $T_\varepsilon(k, \tau) \to \varphi(\tau)$ as the parameters ε_j ($j = 1, 2$) are decreased, the admissible deviation Δ_T limits their maximum possible values. If those values are exceeded, the optimality criterion (1) is not satisfied. On the other hand, the amplitude and frequency of the oscillations of the solution grow with a decrease in the values of ε_j, and this can lead to violation of the control constraints (6). These constraints therefore determine the minimum attainable values of ε_j. Accordingly, any pair of parameters $(\varepsilon_1, \varepsilon_2) \in \Omega = \Omega_1 \cap \Omega_6 \cap \Omega_{14}$ satisfies the stated control problem. But if the set $\Omega = \emptyset$, the problem is uncontrollable under the given conditions.

We illustrate the application of the proposed method in the following example. The heating of a plate is controlled by the convective heat-transfer process (2). It is required to heat the thermally insulated opposite surface, $\psi(\tau) = 0$, from the initial state $f(x) = 0$ to the temperature $0.6T_0$ at the rate ω, i.e., to ensure satisfaction of the condition $\varphi(\tau) = \omega\tau$. The following are specified here: $\Delta_T = 0.1T_0(1 - 0.4\tau)$; $t_1(\tau) = 0$; $t_2(\tau) = \varphi(\tau) + 0.5\ T_0$; $\omega = T_0$.

Figure 3 shows the results of an approximate numerical solution of this problem by a finite-difference procedure [10] using parametric regularization. It is evident from Fig. 3a that the solution $T_\varepsilon(x, \tau)$ does not satisfy the optimality criterion (1) in the first case. With a decrease in ε_1 (Fig. 3b) the deviation Δ_T corresponds to the prescribed error limits, but the value of the parameter ε_1 is close to the minimum admissible for stability (see Fig. 2), and the large oscillations of the control function fail to ensure satisfaction of the constraint (6). The use of two parameters (Fig. 3c) makes it possible for the solution $T_\varepsilon(x, \tau)$ to satisfy the constraints on the error of approximation to the function $\varphi(\tau)$ and on the amplitude of the oscillations of the control function.

NOTATION

x, space coordinate; τ, time; Δ_T, admissible technological error; $T(x, \tau)$, temperature field; H, heat-transfer coefficient; $u(\tau)$, control function; $r_{1,2}$, inside and outside radii of body; $\varepsilon_{1,2}$, regularization parameters; $T_{1,2,3}$, influence functions; μ_n, roots of characteristic equation; Ω_q (q = 1, 6, 14), domain of regularization parameters for satisfaction of condition (q); ω, heating rate.

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V. L. RVACHEV'S QUASI-GREEN'S FUNCTIONS METHOD IN THE THEORY OF HEAT CONDUCTION

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We present a generalization of V. L. Rvachev's method of quasi-Green's functions in connection with the solution of mixed problems for the heat-conduction equation in noncylindrical domains.

Let Ω be a domain in a space of n + 1 dimensions (n = 2, 3), the boundary $\partial\Omega(S_{t_0} + S_{t'} + S_B)$ of which is represented by the normalized equation $\omega(P, t) = 0$, where P is a point with the coordinates $(x_1, x_2, \ldots, x_n)$. We assume that $\omega(P, t)$ is twice continuously differentiable with respect to the spatial coordinates and once continuously differentiable with respect to t; moreover, $\omega(P, t) > 0$ for all $(P, t)\in\Omega/\partial\Omega$ [1].

In the domain Ω we consider the problem of finding a solution of the heat-conduction equation

$$Lu = f\left(L = \Delta - \frac{1}{a^2}\frac{\partial}{\partial t}\right), \tag{1}$$

satisfying the conditions

$$u|_{S_{t'}} = 0, \tag{2}$$

$$u|_{t=t_0} = 0. \tag{3}$$

It was shown in [2] that an arbitrary solution of the heat-conduction equation (1), twice continuously differentiable with respect to $(x_1, \ldots, x_n)$ and continuously differentiable with respect to t, can be represented in the following form:

$$u(P, t) = -a^2\int_{t_0}^{t}\int\overset{(n-1)}{\underset{S_{t'}}{\cdots}}\int\left(v\frac{\partial u}{\partial n'} - u\frac{\partial v}{\partial n'}\right)dS'dt' +$$
$$+\int\overset{(n)}{\underset{G_{t_0}}{\cdots}}\int uv\,d\tau' + \int\overset{(n-1)}{\underset{S_B}{\cdots}}\int uv\cos(n^*, \mathbf{t})\,dS' - a^2\int_{t_0}^{t}\int\overset{(n)}{\underset{G_{t'}}{\cdots}}\int v\,Lu\,d\tau'dt', \tag{4}$$

where

$$v = \delta(P, P', t, t') = \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^n\exp\left(-\frac{r^2}{4a^2(t-t')}\right); \tag{5}$$

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